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GLOBAL COLLOCATION APPROXIMATIONS OF THE FLOW AND TEMPERATURE FIELDS AROUND A GAS AND A VAPOUR BUBBLE

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Abstract—The global collocation method has been applied to potential and viscous flow fields around a bubble. For a rotationally symmetric gas bubble, jet formation, initial acceleration, detachment from a horizontal wall and the motion induced by a surface tension gradient have been studied. For a spherically symmetric vapour bubble implosion and growth have been considered and the bubble radius and temperature are found to oscillate. The computed results have been compared with available experimental data. For a bubble surrounded by an infinitely extended liquid the global collocation method turns out to be particularly useful.

NOMENCLATURE

a, $= k/\rho c$, liquid thermal diffusivity [m^2/s];
A_λ, expansion coefficient, function of λ and t;
B_λ, expansion coefficient, function of λ and t;
c, liquid specific heat at constant pressure [J/kg · K];
C_λ, expansion coefficient, function of λ and t;
D_λ, expansion coefficient, function of λ and t;
D^2, $= \dfrac{\partial^2}{\partial r^2} + \dfrac{1-\mu^2}{r^2}\dfrac{\partial^2}{\partial \mu^2}$ in spherical co-ordinates,
$= \dfrac{\partial^2}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$ in cylindrical co-ordinates [m^{-2}];
D^4, $D^4 = D^2D^2$, fourth order differential operator [m^{-4}];
E_λ, expansion coefficient, function of λ and t;
F, $2(N+1)$—dimensional vector function;
F_λ, expansion coefficient, function of λ and t;
g, strength of gravitational field [m/s^2];
I_0, modified Bessel function of the first kind of order zero with argument x;
I_1, modified Bessel function of the first kind of order one with argument x;
k, liquid thermal conductivity [W/m · K];
K, arbitrary number $< \infty$;
K_0, modified Bessel function of the second kind of order zero with argument x;
K_1, modified Bessel function of the second kind of order one with argument x;
l, latent heat of vaporization [J/kg];
L_0, height of a cylinder [m];
L, non-linear differential operator;
L_1, radius of a cylinder [m];
M, number of collocation points minus one;
$\mathbf{n}$, direction normal to the bubble surface;
$\dfrac{\partial}{\partial n}$, $= \mathbf{n}\cdot\nabla$, derivative in the direction of $\mathbf{n}$ [m^{-1}];
N, number of terms in a truncated series minus one;
p, liquid pressure [Pa];
p_∞, liquid pressure for away from the bubble [Pa];
p_R, liquid pressure on the bubble boundary [Pa];
Pr, $= \nu/a$ liquid Prandtl number;
P_λ, Legendre function of order zero and degree λ with argument x;
Q_λ, Associated Legendre function of order zero and degree λ;
r, radial co-ordinate in spherical and cylindrical co-ordinates [m];
R, bubble radius in spherical and cylindrical co-ordinates [m];
R_0, bubble radius for hemispherical bubbles When $\theta_R(t) = 0$ [m];
t, time [s];
$\mathbf{t}$, direction tangential to the bubble surface;
T, absolute boiling temperature at ambient pressure p_∞ [K];
$\mathbf{u}$, velocity vector [m/s];
u_r, velocity component in r-direction, both in spherical and cylindrical co-ordinates [m/s];
u_ζ, velocity component in ζ-direction, spherical co-ordinates [m/s];
u_z, velocity component in z-direction, cylindrical co-ordinates [m/s];
u_{tr}, translation velocity of a bubble [m/s];
u_{tr_0}, initial translation velocity [m/s];
V, volume of a gas bubble [m^3];

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V_0, initial volume of a gas bubble [m^3];
z, z-co-ordinate, cylindrical co-ordinates [m].

Greek symbols

γ, Poisson's adiabatic constant of the gas;
δ, thermal boundary-layer thickness [m];
ζ, azimuthal angle, spherical co-ordinates;
η, liquid dynamic viscosity [Ns/m^2];
θ, absolute temperature minus T, superheating [K];
θ_∞, superheating at the end of the thermal boundary layer [K];
θ_R, superheating of the bubble boundary [K];
λ, complex exponent in a power series;
μ, $= \cos\zeta$;
ν, $= \eta/\rho$, liquid kinematic viscosity [m^2/s];
ρ, liquid density [kg/m^3];
ρ_2, gas or saturated vapour density [kg/m^3];
σ, surface tension coefficient [N/m];
$\sum$, summation operator;
$\boldsymbol{\tau}$, stress tensor [Pa];
τ_t, $= (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{t}$, tangential stress component on the bubble boundary [Pa];
τ_n, $= (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{n}$, normal stress component on the bubble boundary [Pa];
ϕ, velocity potential in vortex-free flow [m^2/s];
ψ, stream function, in spherical co-ordinates [m^3/s], in cylindrical co-ordinates [m^2/s];
ω, vorticity [s^{-1}] divided by $r\sin\zeta$ in spherical co-ordinates [$m^{-1}\cdot s^{-1}$];
Ω, $= \dfrac{\rho_2}{\rho}\dfrac{\theta_\infty}{T}\dfrac{l}{a}\left(\dfrac{\rho_2 l}{\rho c\theta_\infty}\right)^2$, characteristic frequency describing the transition between initial and asymptotic vapour bubble growth [s^{-1}];
∇, $= \left(\dfrac{\partial}{\partial r}, \dfrac{1}{r}\dfrac{\partial}{\partial\zeta}\right)$ in spherical co-ordinates, $= \left(\dfrac{\partial}{\partial r}, \dfrac{\partial}{\partial z}\right)$ in cylindrical co-ordinates [m^{-1}];
∇^2, $= D^2 + \dfrac{2}{r}\left(\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{\sin\theta}\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\right)$ in spherical co-ordinates, $= D^2 + \dfrac{2}{r}\dfrac{\partial}{\partial r}$ in cylindrical co-ordinates [m^{-2}].

Subscripts

i, number of a collocation point;
k, number of a term in a series;
tr, translation;
R, on the bubble wall in the liquid;
λ, complex subscript of an expansion coefficient;
0, initial, at $t = 0$;
∞, far away from the bubble.

Superscripts

differentiation with respect to the argument;
total differentiation with respect to time;
approximate value.

1. INTRODUCTION

THE BEHAVIOUR of bubbles has been the subject of many investigations since direct contact between gas and liquid often occurs in industrial processes, especially in two phase flow and boiling heat transfer. For the qualitative understanding of the complex physical phenomena involved, mathematical simulation is an indispensable tool. Theoretically spoken, the gas in the bubbles and the surrounding liquid can adequately be described by the Navier–Stokes equations and also the appropriate boundary conditions can easily be formulated. So, with prescribed initial conditions the evolution in time of the system is defined. However, even for one bubble with uniform interior and a laminar flow field around it, there is a formidable lack of exact solutions, due both to the non-linearities in the Navier–Stokes equations and to the *a priori* unknown position and irregular shape of the boundary.

So we are forced to use approximation methods. By using the powerful method of uniform asymptotic expansions, other approximation methods must be applied to solve the resulting equations of the expansion hierarchy. So it makes sense to consider numerical approximation methods, i.e. methods which map the space–time continuum into a discrete space–time mesh.

Also the problems in numerical computation are big. Modern high speed computers with large memory capacity make certain problems more accessible to numerical methods, but they do not give an answer to questions like stability, convergence and error bounds. A pragmatic approach is adopted here: the criterion of computational efficiency has been used to choose the global collocation method and the approximate results are justified by comparison with available experimental data.

All calculations were performed on a Burroughs B6700; the computation time varied from 1 to 30 min.

2. MATHEMATICAL FORMULATION

Both the equations of motion for an incompressible liquid surrounding one bubble and the necessary auxiliary conditions are presented in spherical co-ordinates, assuming rotational symmetry. The equations of motion of the gas in the bubble are assumed to reduce to the condition that the pressure in the gas is homogeneous.

It is of great advantage to satisfy the continuity equation by the introduction of a stream function ψ, so that;

$$\mathbf{u} = (u_r, u_\zeta) = \frac{1}{r\sin\zeta}\left(-\frac{1}{r}\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial r}\right)\psi. \tag{1}$$

The momentum equations in the r-and ζ-direction are:

$$\nabla(p+\rho g r\cos\zeta)=\frac{\eta}{r\sin\zeta}\left(-\frac{1}{r}\frac{\partial}{\partial\zeta},\frac{\partial}{\partial r}\right)D^2\psi$$

$$-\rho\left(\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\zeta}{r}\frac{\partial}{\partial\zeta}\right)\mathbf{u}-\rho\frac{u_\zeta}{r}(-u_\zeta,u_r). \quad (2)$$

By taking the curl of equation (2), pressure and gravity can be eliminated, resulting in the vorticity diffusion equation (3) and a Poisson equation for ψ (4):

$$\rho\left(\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\zeta}{r}\frac{\partial}{\partial\zeta}\right)\omega=\frac{\eta}{r^2\sin^2\zeta}D^2(\omega r^2\sin^2\zeta), \quad (3)$$

$$D^2\psi=\omega r^2\sin^2\zeta. \quad (4)$$

where ω is the vorticity divided by $r\sin\zeta$.

Finally, neglecting viscous dissipation, the diffusion equation for the thermal energy is:

$$\rho c\left(\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\zeta}{r}\frac{\partial}{\partial\zeta}\right)\theta=k\nabla^2\theta. \quad (5)$$

In order to obtain a well-posed partial differential problem, capable of predicting the evolution in time of both the flow- and temperature fields around the bubble and of the co-ordinates of the bubble wall, four boundary conditions must be presented.

One condition for the normal stress and one condition for the tangential stress for equations (3) and (4); one condition for the temperature or heat flux for equation (5) and one relation between the normal flow velocity and the bubble wall displacement to describe the position and shape of the bubble boundary.

For non-translating, nearly spherically symmetric bubbles and for translating bubbles during a short time after the onset of motion, the influence of the tangential stress condition is restricted to a thin boundary layer around the bubble, cf. Batchelor [1].

Outside this layer the solution of (3) and (4) is $\omega=0$ and equation (4) can be used to define a velocity potential ϕ, so that:

$$\mathbf{u}=\nabla\phi, \quad (6)$$

$$\nabla^2\phi=0. \quad (7)$$

For potential flow the momentum equations (2) can be integrated, yielding the Bernoulli equation for the pressure:

$$p+\rho g r\cos\zeta=-\rho\left(\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\nabla\phi\cdot\nabla\phi\right)+p_\infty(t). \quad (8)$$

In the case of viscous flow, where inertia effects are neglected, the equations (3, 4) reduce to:

$$D^4\psi=0. \quad (9)$$

Since exact solutions of (9) are well known, the pressure can easily be found by integration of (2), neglecting the inertia terms. General solutions of equation (4) with $\omega=0$ are cf. Abramowitz and Stegun [2]:

$$\psi(r,\zeta,t)=r^\lambda\sin^2\zeta\{A_\lambda(t)P'_{\lambda-1}(\cos\zeta)+B_\lambda(t)Q'_{\lambda-1}(\cos\zeta)\}. \quad (10)$$

For solutions of equation (9) one has to add:

$$\psi(r,\zeta,t)=r^\lambda\sin^2\zeta\{C_\lambda(t)P'_{\lambda-3}(\cos\zeta)+D_\lambda(t)Q'_{\lambda-3}(\cos\zeta)\}. \quad (11)$$

A_λ, B_λ, C_λ, D_λ are functions of t depending on the complex number λ, $P_\lambda(\mu)$ and $Q_\lambda(\mu)$ are Legendre functions, respectively associated Legendre functions of order zero and degree λ. The prime means differentiation with respect to the arguments.

If $\lambda=0$, $\lambda=1$ then $P'_{\lambda-1}(\mu)=0$; $f\lambda=2$, $\lambda=3$ then $P'_{\lambda-3}(\mu)=0$; in these degenerate cases the following independent solution must be added:

$$\psi(r,\zeta,t)=E_\lambda(t)\cos\zeta. \quad (12)$$

3. THE GLOBAL COLLOCATION METHOD

In potential and viscous flow situations general solutions for the stream function are known and these solutions can be matched to the boundary conditions by an appropriate choice of the expansion coefficients $A_\lambda(t),\ldots E_\lambda(t)$, cf. Section 2.

In most cases this has to be done numerically by a method of weighted residuals, cf. Finlayson [3]. For reasons of computational efficiency the collocation method is preferable. In this method the boundary is discretized into a finite mesh of so called collocation points and only there the boundary conditions are applied. In this way a linear system of algebraic equations is obtained.

Assume that a boundary condition has the following form:

$$\left[L\left\{\sum_{k=0}^{\infty}a_k(t)f_k(r,\mu)\right\}\right]_{r=R(\mu,t)}=0. \quad (13)$$

Herein $\{f_k(r,\mu)\}_{k=0}^{\infty}$ is the general solution and $\{a_k(t)\}_{k=0}^{\infty}$ are the coefficients to be fitted.

Applying collocation on the points $\{R(\mu_i,t),\mu_i\}_{i=0}^{N}$, an approximate solution with a finite number of expansion coefficients $\{\dot{a}_k(t)\}_{k=0}^{N}$ is obtained.

From interpolation theory it is known that:

$$\left|L\left\{\sum_{k=0}^{N}\dot{a}_k(t)f_k(r,\mu)\right\}\right|_{r=R(\mu,t)}<K\cdot\frac{(\mu-\mu_0)(\mu-\mu_1)\ldots(\mu-\mu_i)\ldots(\mu-\mu_N)}{(N+1)!}. \quad (14)$$

Convergence for $N\to\infty$ can be guaranteed if $\{\mu_i\}_{i=0}^{N}$ are the zeros of ultra spherical polynomials, e.g. the Chebyshev and the Legendre polynomial, cf. Fox and Parker [4].

For the diffusion equation (5) a general solution cannot be given; in that case an appropriately chosen trial function with adjustable coefficients is substituted into the equation and boundary conditions. Now collocation points in the whole temperature field are needed. In order to avoid ill conditioned matrices the number of collocation points must be low, so the solution has only a global character.

4. SPHERICALLY SYMMETRIC VAPOUR BUBBLES

In spherically symmetric flow fields the Bernoulli equation (8), combined with the linearized Clapeyron equation, cf. van Stralen [5] results in the Rayleigh equation:

$$R\ddot{R}+\tfrac{3}{2}\dot{R}^2 = \frac{\rho_2 l}{\rho T}\theta_R - \frac{2\sigma}{\rho}\frac{1}{R} - \frac{4\eta}{\rho}\frac{\dot{R}}{R}. \tag{15}$$

During growth the bubble fills up with vapour, which requires evaporation of liquid at the bubble boundary. The required heat flux is given by:

$$\left(k\frac{\partial\theta}{\partial r}\right)_{r=R} = \rho_2 l\dot{R}. \tag{16}$$

In order to solve the heat balance equation (5), with heat flux requirement (16) as boundary condition, a simple form of the collocation method is used.

The temperature in the thermal boundary layer around the bubble is expanded in:

$$\theta(r,t) = \theta_\infty + (\theta_R(t)-\theta_\infty)\exp\left(-\frac{r-R(t)}{\delta(t)}\right), \tag{17}$$

herein $\theta_R(t)$ and $\delta(t)$ are the coefficients to be adjusted. Substitution of (17) into (16) and into (5) on the collocation point $r = R(t)$ results in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\theta_R-\theta_\infty}{\theta_\infty}\right)^2 = \frac{2}{a}\left(\frac{\rho_2 l}{\rho c\theta_\infty}\right)^2\dot{R}^2. \tag{18}$$

By taking the square root the negative sign must be used and at the time where $\dot{R}$ changes sign θ_R is made discontinuously equal to θ_∞; in this way the physically unrealistic growth with increasing temperature and implosion with decreasing temperature are avoided.

The coupled system of ordinary differential equations (15, 18) can be solved numerically, e.g. by a Runge Kutta method. Figure 1 presents both computed and experimental results for a growing bubble. It is noted that the calculations are performed for an initially uniformly superheated liquid, whereas the experiment represents growth in a temperature gradient at a superheated wall.

After a certain time of growth a stage is reached where $\theta_R(t) \simeq 0$ and inaccuracies in the computation of θ_R will lead to artificial oscillation or stagnation, as can be seen from (15). Hence on that timescale another procedure must be used. It has been shown by Scriven [6] that for $\theta_R(t) = 0$ the bubble has the following growth rate:

$$R_0(t) = 2\left(\frac{\rho c\theta_\infty}{\rho_2 l}\right)(at)^{1/2}. \tag{19}$$

In order to incorporate the effect of inertia a linearized solution of system (15, 18) can be found when R is expanded in:

$$R(t) = R_0(t)\{1+\varepsilon(t)\}, \quad |\varepsilon(t)| \ll 1. \tag{20}$$

Substitution of (20) into (15, 18) leads to:

$$t\ddot{\varepsilon}+\tfrac{7}{2}\dot{\varepsilon}+\Omega\varepsilon = 0 \quad \text{for} \quad t \gg \Omega^{-1}. \tag{21}$$

$$\Omega = \frac{\rho_2}{\rho}\frac{\theta_\infty}{T}\frac{l}{a}\left(\frac{\rho_2 l}{\rho c\theta_\infty}\right)^2. \tag{22}$$

An approximate solution of (21) is:

$$\varepsilon(t) = A\sin\{2(\Omega t)^{1/2}\}. \tag{23}$$

Physically, the oscillations are obtained as follows: when the heat flux to the bubble is just enough for the required heat of evaporation the bubble growth rate equals $R_0(t)$.

If for some reason the growth is more rapid, more

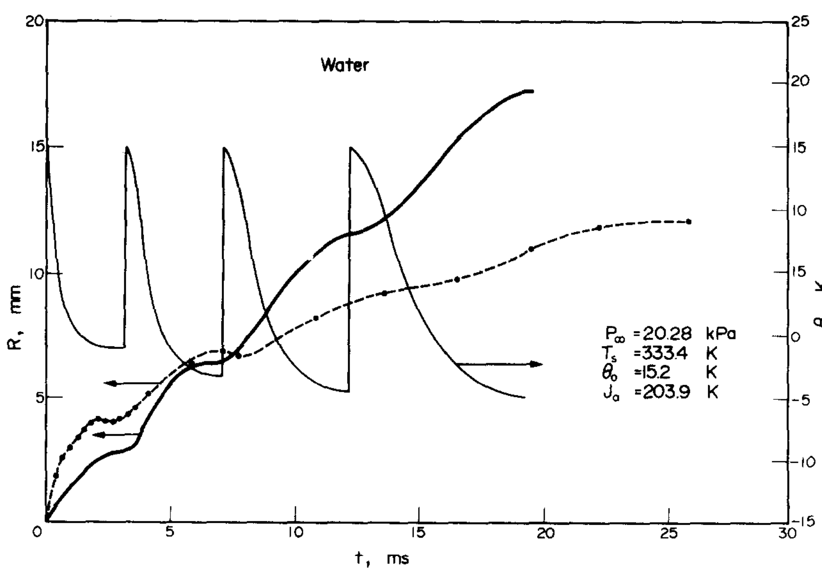

FIG. 1. Experimental (▲) and theoretical growth of a free vapour bubble in water ($\theta_\infty = \theta_0 = 14.6$ K, $p_\infty = 26.72$ kPa, computation time 1 min).

heat of evaporation is required and the temperature of the liquid at the bubble boundary will decrease. According to equation (15) a lower temperature will decrease the growth rate, consequently less heat of evaporation is required and the temperature increases. Due to inertia the liquid velocity lags behind the change in temperature and so oscillations occur.

The amplitude A depends on the induced disturbances, e.g. caused by neighbouring bubbles; the discontinuity in θ_R, introduced to solve the system (15, 18), represents an artificial amplitude, but note that the frequency agrees quantitatively with the experimentally observed values. In (21, 23), the damping is not well represented; in [7], Zijl gives arguments that the damping may be represented by $\exp(-\Omega t)$. Introduction of an artificial amplitude, in such a way that solution (20, 23) matches the solution for initial isobaric growth, cf. Van Stralen [5], results in:

$$R(t) = \left\{(at)^{1/2} - \tfrac{1}{2}\left(\frac{a}{\Omega}\right)^{1/2} \sin\{2(\Omega t)^{1/2}\} + \tfrac{1}{5}(at)^{1/2} \sin\{2(\Omega t)^{1/2}\}\, e^{-\Omega t}\right\} \cdot 2\frac{\rho c \theta_\infty}{\rho_2 l}, \qquad 0 \leqslant t < \infty. \quad (24)$$

In the case that the Jakob number $Ja = \rho c \theta_\infty / \rho_2 l < 1$, no thin thermal boundary layer exists. Now the temperature is expanded in a polynomial in $1/r$. The choice of a parabola, collocated at $r = R$ results in:

$$\dot{\theta}_R = -2a \cdot \frac{\theta_R - \theta_\infty}{R^2} - 2\frac{\theta_\infty}{Ja} \cdot \frac{\dot{R}}{R}. \quad (25)$$

Expression (25) represents a better approximation than equation (18), since in (25) no discontinuity in the temperature needs to be introduced and in the asymptotic stage $t \to \infty$, $\dot{R} \sim 1/t^{1/2}$, $\theta_R \sim 1/t$, the solution approaches the exact solution derived by Scriven [6]. In the same way as for the derivation of (24) the bubble radius can be approximated by:

$$R(t) = \left\{(at)^{1/2} - \left(\frac{aJa}{2\Omega}\right)^{1/2} \sin\left\{\left(\frac{2\Omega t}{\mathrm{Ja}}\right)^{1/2}\right\} + \frac{1}{\sqrt{(6)}} Ja \sin\left\{\left(\frac{2\Omega t}{Ja}\right)^{1/2}\right\} e^{-\frac{\Omega t}{2Ja}}\right\} \cdot 2Ja, \qquad 0 \leqslant t < \infty. \quad (26)$$

In Fig. 2, a cubic polynomial with collocation points on $r = R$ and $r = R(1+0.01)$ is used to calculate an imploding bubble, also experimental values are presented.

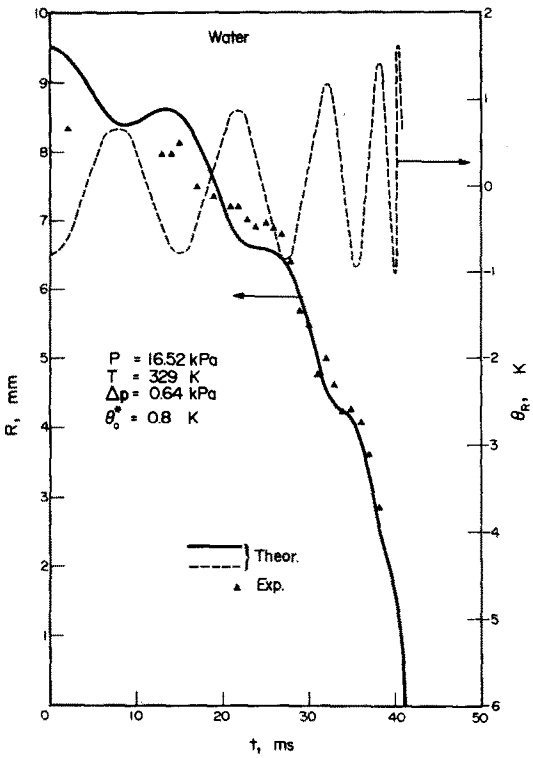

FIG. 2. Experimental (▲) and theoretical implosion of a free vapour bubble in water ($\theta_\infty = \theta_0 = -0.8$ K, $p_\infty = 16.52$ kPa, computation time 1 min).

5. ROTATIONALLY SYMMETRIC GAS BUBBLES

From equations (10, 12) it can be concluded that the solution of equation (6) with zero velocity in $r \to \infty$ and without singularity in $\zeta = 0$ has the following form:

$$\phi(r, \zeta, t) = \sum_{k=0}^{\infty} a_k(t) \frac{1}{r^{k+1}} P_k(\cos\zeta). \quad (27)$$

The co-ordinates of the bubble boundary $r = R(\zeta, t)$ are also expanded in Legendre polynomials:

$$R(\zeta, t) = \sum_{k=0}^{\infty} b_k(t) P_k(\cos\zeta). \quad (28)$$

In order to apply the collocation method, the series (31, 32) are truncated after $N+1$ terms and also $N+1$ collocation angles ζ_i are chosen. Hence, one has:

$$R_i(t) = R(\zeta_i, t) = \sum_{k=0}^{N} [P_k(\cos\zeta_i)] b_k(t), \quad (29)$$

$$\phi_i(t) = \phi(R_i(t), \zeta_i, t) = \sum_{k=0}^{N} \left[\left(\frac{1}{R_i(t)}\right)^{k+1} P_k(\cos\zeta_i)\right] a_k(t). \quad (30)$$

With equation (8) the normal stress condition on the bubble boundary can be derived, yielding:

$$\frac{\partial\phi}{\partial t} = -\tfrac{1}{2}\nabla\phi\cdot\nabla\phi - gR\cos\zeta - \frac{p_0}{\rho}\left(\frac{V_0}{V(t)}\right)^{\gamma} + \frac{p_\infty}{\rho} - \frac{\sigma}{\rho}\frac{1}{R}\left[\frac{1 + 2\left(\frac{1}{R}\frac{\partial R}{\partial\zeta}\right)^2 - \frac{1}{R}\frac{\partial^2 R}{\partial\zeta^2}}{\left\{1 + \left(\frac{1}{R}\frac{\partial R}{\partial\zeta}\right)^2\right\}^{3/2}} + \frac{1 - \frac{1}{R}\frac{\partial R}{\partial\zeta}\cdot\frac{\cos\zeta}{\sin\zeta}}{\left\{1 + \left(\frac{1}{R}\frac{\partial R}{\partial\zeta}\right)^2\right\}^{1/2}}\right]. \quad (31)$$

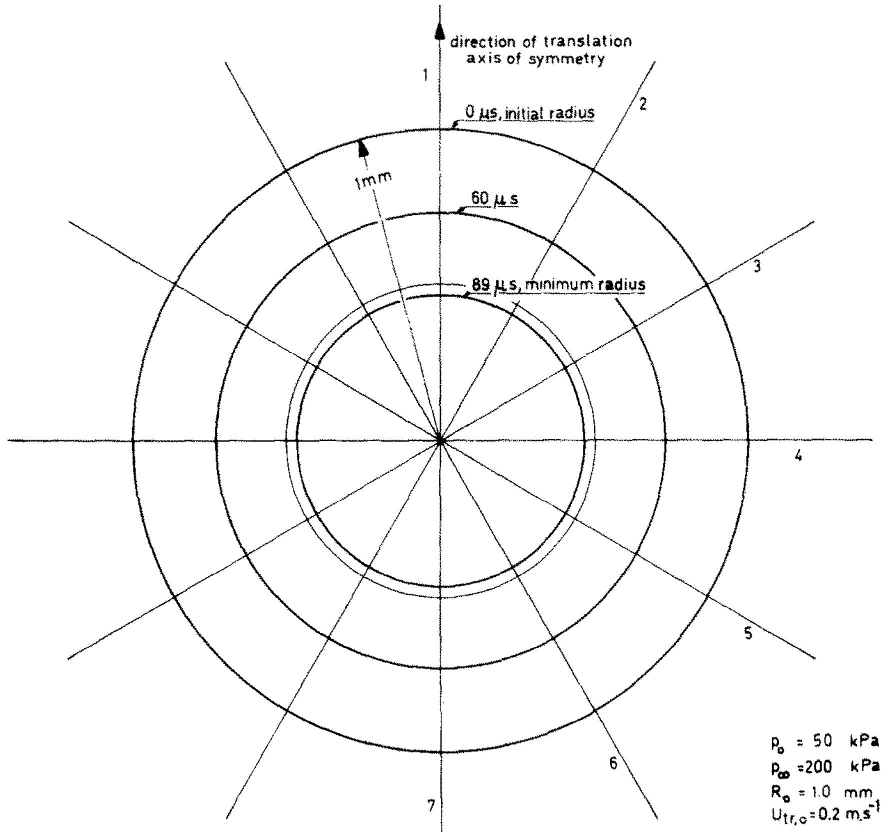

FIG. 3(a). Imploding phase of a translating gas bubble, hit by a pressure step ($N+1 = 7$, $p_0 = 50\,\mathrm{kPa}$, $p_\infty = 200\,\mathrm{kPa}$, $R_0 = 1\,\mathrm{mm}$, $u_{tr_0} = 0.2\,\mathrm{m/s}$, computation time 5 min).

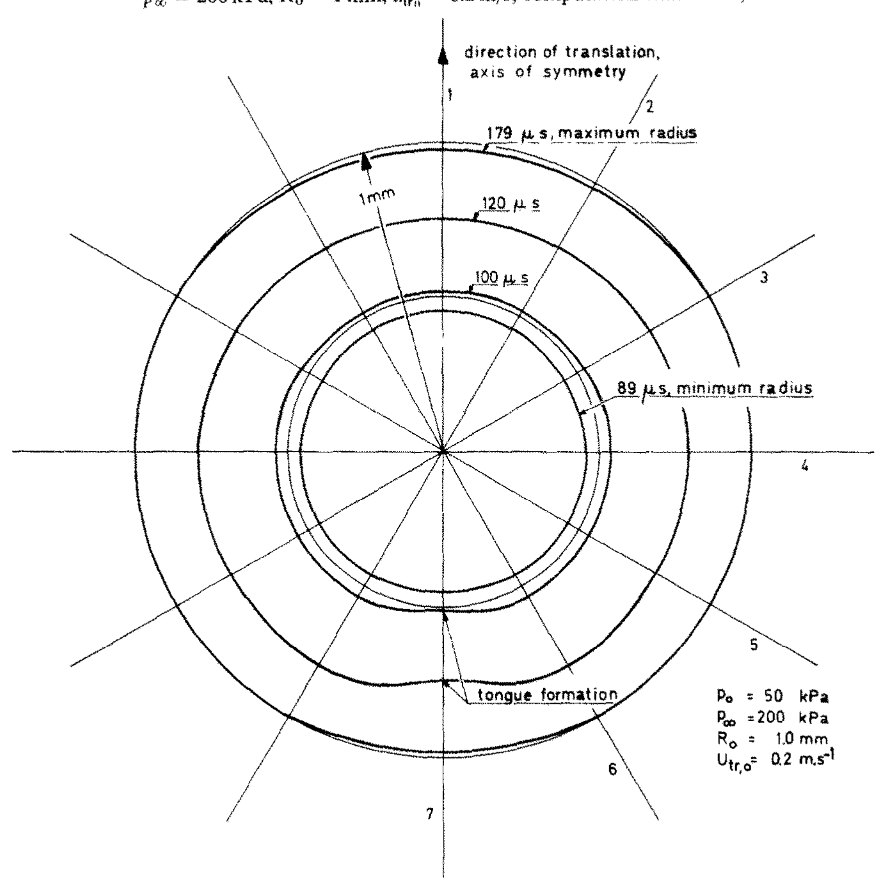

FIG. 3(b). Growing phase of the bubble from Fig. 3(a) (computation time 5 min).

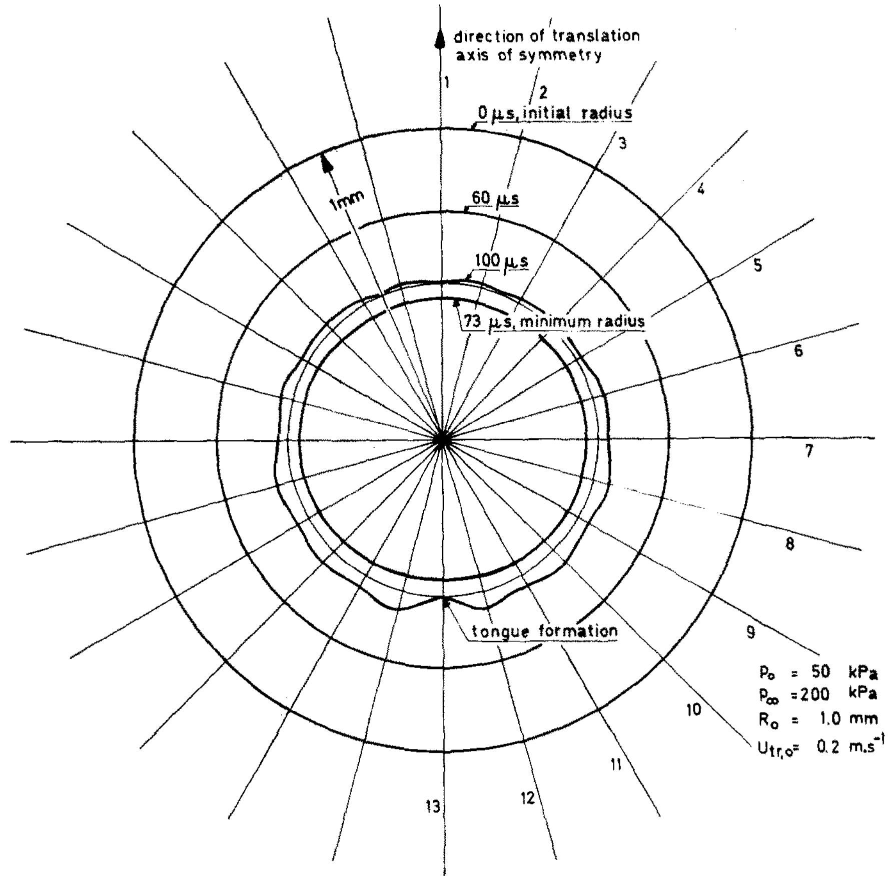

FIG. 4. The bubble from Figs. 3(a), 3(b), with $N+1 = 13$ (computation time 10 min).

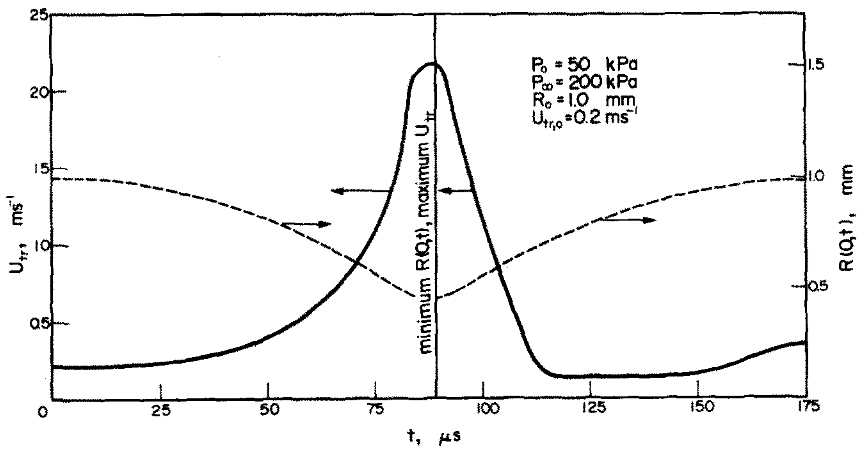

FIG. 5. R_0 and U as a function of t for the bubble of Figs. 3(a) and 5(b).

The condition that the normal flow velocity at the bubble boundary equals the bubble wall velocity leads to:

$$\frac{\partial R}{\partial t} = \frac{\partial \phi}{\partial r} - \frac{1}{R^2}\frac{\partial \phi}{\partial \zeta}\frac{\partial R}{\partial \zeta}. \tag{32}$$

With equations (31, 32) a system of non-linear coupled ordinary differential equations is derived:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} R_i \\ \phi_i \end{pmatrix} = F\{(R_j, \phi_j)_{j=0}^{N}\}. \tag{33}$$

Numerically spoken the vector function F is a procedure in which $\{\dot{R}_i, \dot{\phi}_i\}_{i=0}^{N}$ are evaluated from given values of $\{R_j, \phi_j\}_{j=0}^{N}$ and for every timestep the matrix equations (29, 30) have to be solved. Since these matrices are full, the number of collocation points may not be too high and special attention must be given to the scaling of (30). The advantage of this method is that for $N+1 = 1$ the system describes a spherically symmetric bubble.

With the methods described in Sections 4 and 5 also the behaviour of a rotationally symmetric vapour bubble can be described, this has been performed by Joosten, Zijl and Van Stralen [8]. In the following subsections a number of examples are given; the

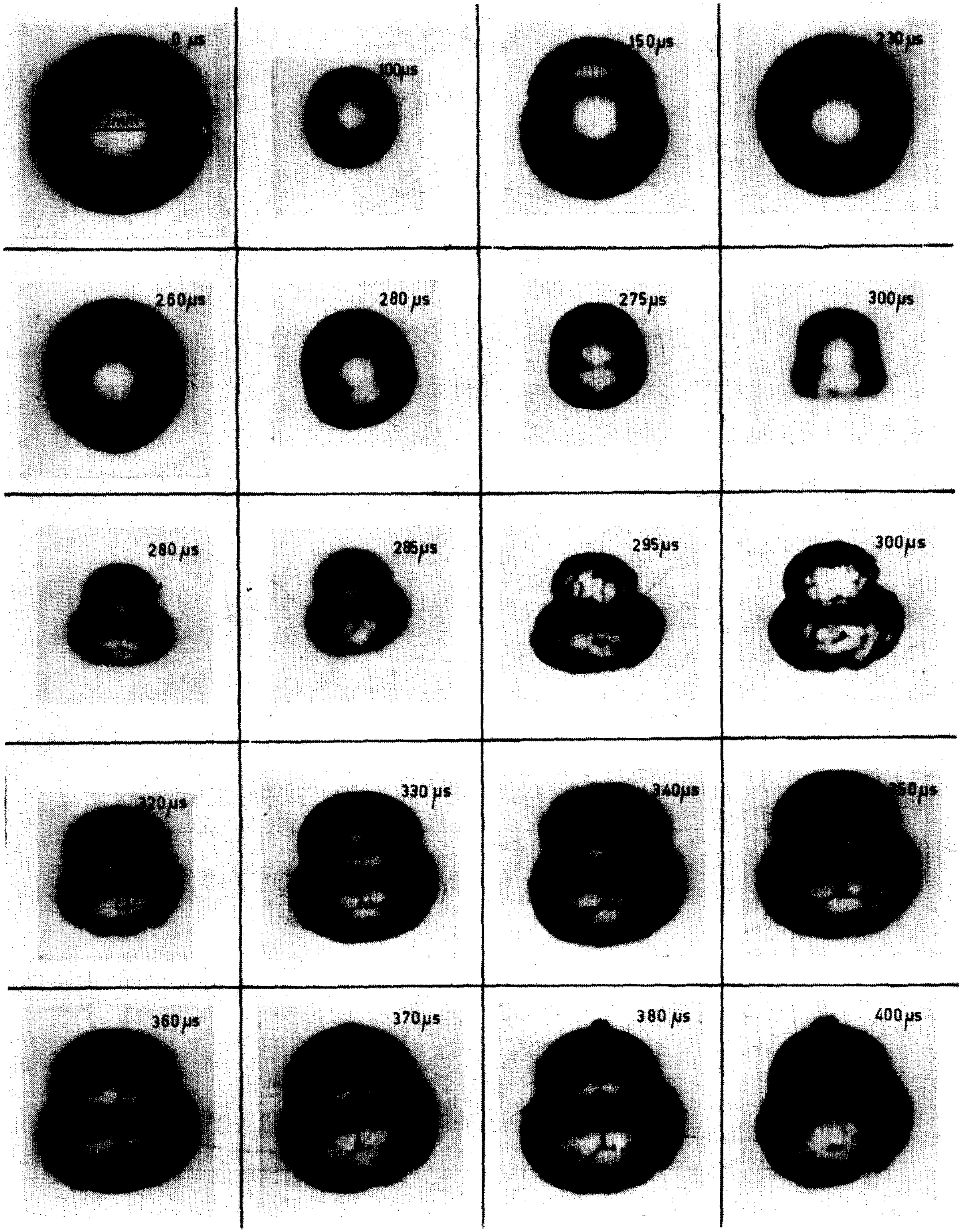

FIG. 6. Gas bubble in water, hit by a pressure step in a shock tube. ($P_0 = 75\,\text{kPa}$, $P_\infty = 250\,\text{kPa}$, $R_0 = 1\,\text{mm}$, $U_{tr_0} = 0.04\,\text{m/s}$).

collocation angles are chosen equidistantly in the interval $-\frac{1}{2}\pi \leqslant \zeta_i \leqslant \frac{1}{2}\pi$.

5.1. *Stable and unstable phases for a translating gas bubble, hit by a pressure step*

It is assumed that on $t = 0$ a bubble is formed with spherical shape, $R_0 = 1$ mm and internal pressure $p_0 = 50$ kPa; at the same time the pressure in the liquid is $p_\infty = 200$ kPa.

The initial translation velocity $U_{tr_0} = 0.2$ m/s and gravity is neglected. The bubble will start to implode, this is presented in Figs. 3(a) and 4. From Fig. 5 it is observed that an imploding bubble is accelerated. During implosion a high pressure is built up in the gas and after a certain time the bubble will start growing. This phase is shown in Figs. 3(b) and 4 and it is seen that formation of a liquid tongue at the rear of the bubble occurs. In Fig. 5 it is shown that a growing bubble is decelerated, a phenomenon which is also important in calculations of the adherence time of a vapour bubble, growing on a superheated wall. Figure 6 presents a measured bubble, generated in a shock

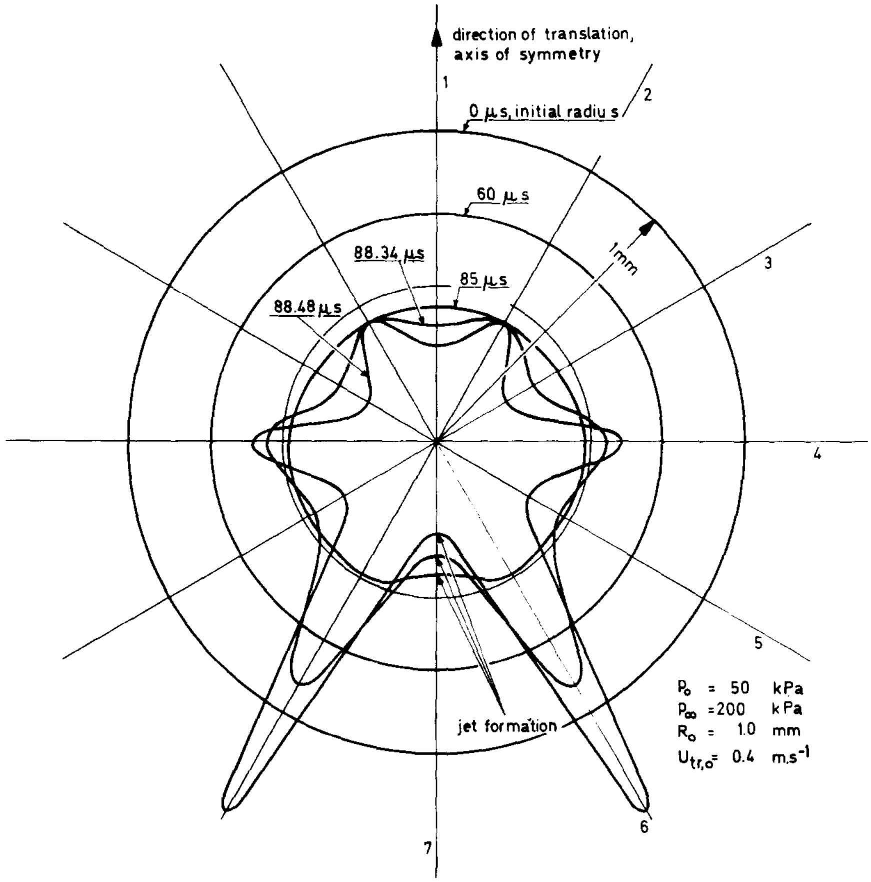

FIG. 7. Jet formation; the bubble of Figs. 3(a) and (b) but with an initial translation velocity $U_0 = 0.4$ m/s (computation time 20 min).

tube. In the instable growing phase the agreement is only qualitative in predicting tongue formation, cf. Section 5.2.

5.2. *Jet formation at the rear of a translating gas bubble, hit by a pressure step*

In the unstable situation small deviations from the exact solution may blow up in time and the collocation method is likely to fail. Nevertheless, Fig. 7 shows the same gas bubble of 5.1, but now with an initial translation velocity $U_{\mathrm{tr}_0} = 0.4$ m/s, which is the cause of jet formation. The result agrees qualitatively with the observations of Fig. 6.

5.3. *Initial acceleration of a gas bubble*

Figure 8 shows a gas bubble which is formed on $t = 0$ without initial velocity, with radius $R_0 = 1$ cm in a gravitational field of $g = 10\,\mathrm{m/s^2}$. From the calculations it follows that the initial acceleration equals $20\,\mathrm{m/s^2}$ and after a longer time tongue formation occurs. The results agree with those of Walters and Davidson [9].

5.4. *Adherence of a gas bubble at a smooth wall*

Figure 9 shows a smooth wall and at $t = 0$ both above and beneath it a bubble of radius $R_0 = 1$ cm is formed in a gravitational field with $g = 10\,\mathrm{m/s^2}$. For $t > 0$ contraction and smearing out is shown.

In order to obtain a zero normal velocity on the fixed wall only even Legendre polynomials were used.

6. A SURFACE TENSION DRIVEN CAVITY

A gradient in surface tension gives rise to a tangential stress in the liquid adjacent to the bubble boundary, in such a way that the liquid tends to move along the boundary in the direction of the gradient, cf. Levich [10], or:

$$(\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{t} = -(\nabla \sigma \cdot \mathbf{t}). \tag{34}$$

The reasons for surface tension gradients may be either gradients in temperature, gradients in concentration of surface substances, or the presence of a varying electric charge on the interfacial layer between gas and liquid.

In order to show the effect of boundary condition (34) a rotationally symmetric gas bubble inside a cylindrical cavity is considered, cf. Fig. 10.

The cylinder wall $r = L_1$ and the bubble boundary $r = R(z, t)$ are assumed to be adiabatic and the upper wall has a higher temperature $\theta(r, 0, t) = \theta_0$ than the lower wall, at which $\theta(r, L_0, t) = 0$. In this way free convection cannot occur and if gravity and excess pressure effects are neglected motion is only induced by a surface tension gradient.

A solution for viscous flow is presented. The equations of motion are reformulated for cylindrical co-

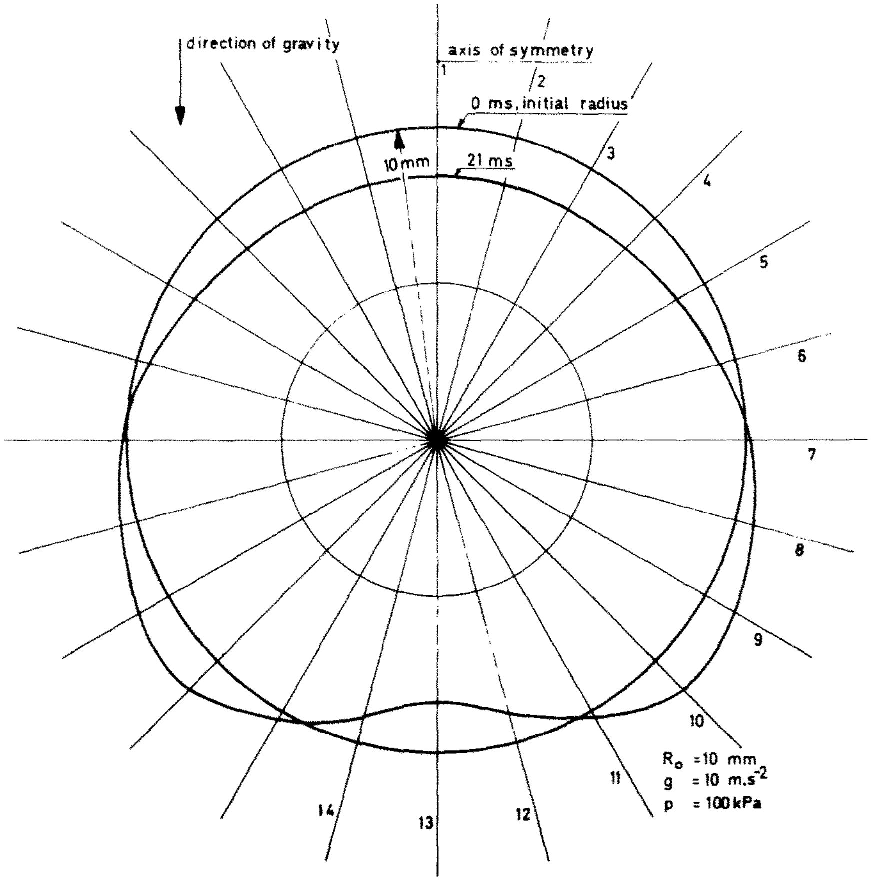

FIG. 8. Initial acceleration of a gas bubble in the gravity field $g = 10\,\mathrm{m/s}$, $R_0 = 1\,\mathrm{cm}$, $P_0 = P_\infty = 100\,\mathrm{kPa}$, computation time (15 min).

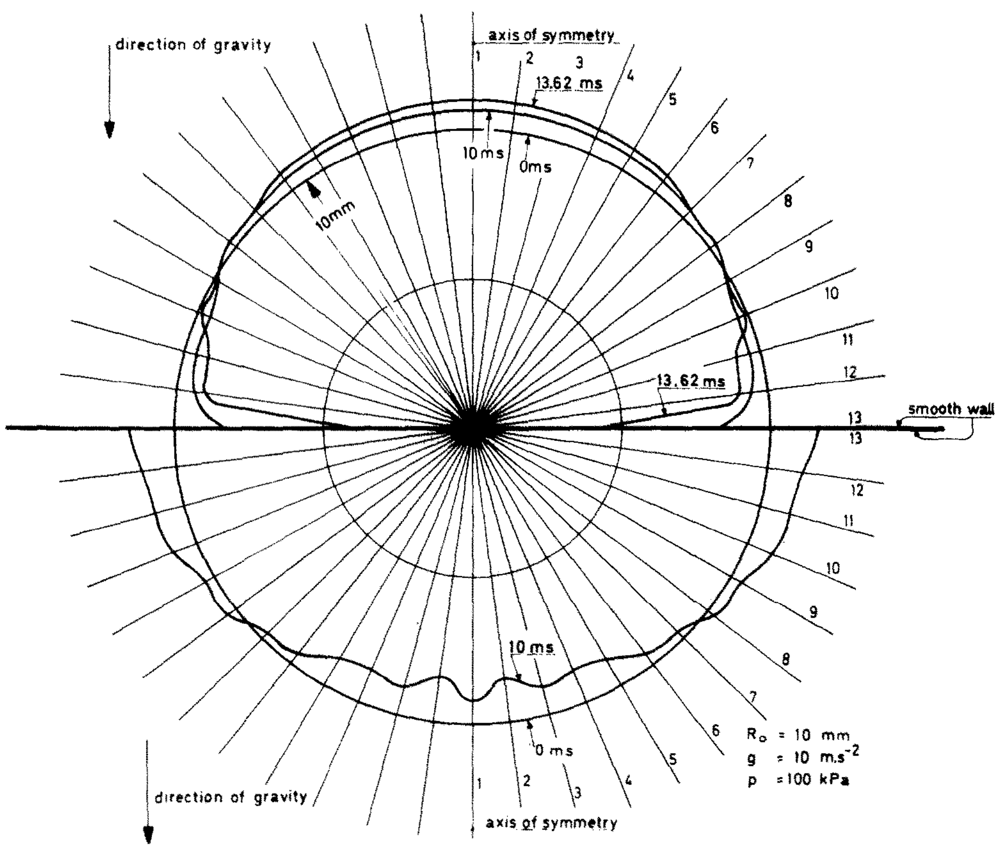

FIG. 9. Gas bubble on a wall in the gravity field $g = 10\,\mathrm{m/s}$, $R_0 = 1\,\mathrm{cm}$, $P_0 = P_\infty = 100\,\mathrm{kPa}$ (computation time 2 min).

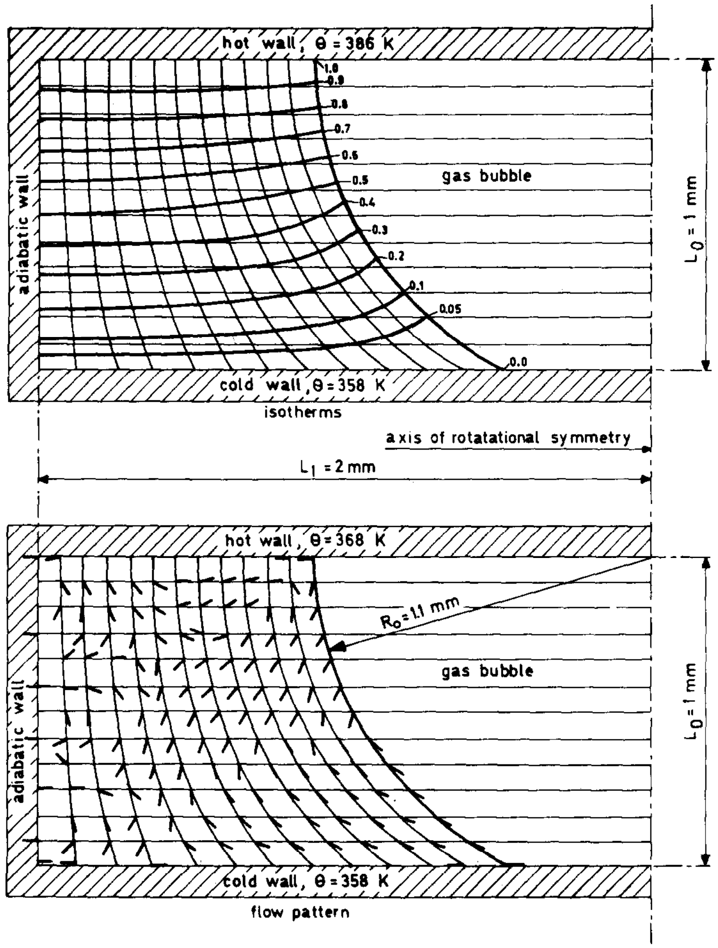

FIG. 10. Gas bubble and water in a heated cylinder ($N+1 = 5$, $M+1 = 11$, computation time 2 min).

ordinates, yielding:

$$\mathbf{u} = (u_r, u_z) = \frac{1}{r}\left(-\frac{\partial}{\partial z}, \frac{\partial}{\partial r}\right)\psi, \tag{35}$$

$$D^4\psi = 0. \tag{36}$$

$$\nabla(p+\rho g z) = \frac{\eta}{r}\left(-\frac{\partial}{\partial z}, \frac{\partial}{\partial r}\right)D^2\psi, \tag{37}$$

$$\nabla^2\theta = 0. \tag{38}$$

The general solution of (38), satisfying the boundary conditions at the heated upper and lower wall is:

$$\theta(r,z,t) = \sum_{k=1}^{\infty} \{A_k(t)I_0(\lambda_k r) + B_k(t)K_0(\lambda_k r)\}\sin(\lambda_k z) + \left(1 - \frac{z}{L_0}\right)\theta_0, \tag{39}$$

where

$$\lambda_k = \frac{\pi \cdot k}{L_0}. \tag{40}$$

Determination of $A_k(t)$, $B_k(t)$ must be performed by matching of solution (39, 40) to the boundary condition $\partial\theta/\partial n = 0$ on the cylinder wall $r = L_1$ and on the bubble boundary. This has been done by least squares collocation. Series (39) is truncated after $k = N$ and on each wall M collocation points are chosen so that $M \geqslant N$. Satisfying the boundary conditions on the $2M$ collocation points results in $2M$ linear algebraic equations for the $2N$ coefficients. Such a system can be solved numerically with a least squares procedure. Isotherms for $N = 5$ are presented in Fig. 10(a). After having determined the temperature field the tangential stress condition for (39, 40) is known (in water at normal conditions $d\sigma/d\theta \simeq -0.2$ mN/m · K).

The general solution of (36), satisfying the condition of zero normal velocity at the upper and lower wall is:

$$\psi(r,z,t) = \sum_{k=1}^{\infty} \{A_k(t)rI_1(\lambda_k r) + B_k(t)rzI_1(\lambda_k r) + C_k(t)r^2I_0(\lambda_k r) + D_k(t)rK_1(\lambda_k r) + E_k(t)rzK_1(\lambda_k r) + F_k(t)r^2K_0(\lambda_k r)\}\sin(\lambda_k z). \tag{41}$$

From equation (37) the pressure can be determined as a function of the expansion coefficients $Ak(t), \dots F_k(t)$ and so the remaining boundary conditions can be satisfied by a suitable choice of these coefficients. This has again been done by least squares collocation and Fig. 10(b) presents the flow field around a prescribed spherical bubble shape; it can be seen that the bubble tends to flatten.

In contrast to the situation described in Section 5, use of only one collocation point obviously produces nonsense, whereas too much collocation points (here $N > 5$) lead to numerically singular matrices. Due to the fact that the pressure occurs explicitly in the normal stress boundary condition, usual finite difference or element methods are hardly applicable. It might be that local collocation with suitably chosen splines is the right answer to such problems.

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APPROXIMATION OBTENUE PAR COLLOCATION GLOBALE DES CHAMPS DE VITESSE ET DE TEMPERATURE AUTOUR D'UNE BULLE DE GAZ OU DE VAPEUR

Résumé—La méthode de collocation globale a été appliquée à l'étude des écoulements potentiels et visqueux autour d'une bulle. On a étudié, dans le cas d'une bulle gazeuse à symétrie sphérique, la formation du jet, l'accélération initiale, le détachement de la paroi horizontale et le mouvement induit par un gradient de tension superficielle. Dans le cas d'une bulle de vapeur à symétrie sphérique, on a étudié l'implosion et la croissance, on trouve une variation oscillatoire du rayon et de la température des bulles. Les résultats du calcul ont été comparés aux données expérimentales disponibles. Dans le cas d'une bulle entourée d'un liquide s'étendant à l'infini la méthode de collocation globale se trouve être particulièrement utile.

NÄHERUNGSWEISE BESTIMMUNG DES STRÖMUNGS- UND TEMPERATURFELDES UM EINE GAS- UND EINE DAMPFBLASE MIT HILFE DER METHODE DER GLOBALEN KOLLOKATION

Zusammenfassung—Die Methode der globalen Kollokation wurde auf Potentialströmungen und Zähigkeitsströmungen um Blasen angewandt. Für eine rotationssymmetrische Gasblase wurde die Strahlbildung, die Anfangsbeschleunigung, die Ablösung von einer horizontalen Wand und die durch Oberflächenspannungsgradienten hervorgerufene Strömung untersucht. Für eine kugelförmige Dampfblase wurde der Zerfalls- und Wachstumsvorgang betrachtet; sowohl beim Blasenradius wie bei der Temperatur ergaben sich dabei Oszillationen. Die berechneten Werte wurden mit verfügbaren experimentellen Daten verglichen. Die Methode der globalen Kollokation erwies sich als besonders nützlich für eine Blase in einer unendlich ausgedehnten Flüssigkeit.

ГЛОБАЛЬНЫЕ КОЛЛОКАЦИОННЫЕ АППРОКСИМАЦИИ ПОЛЕЙ СКОРОСТЕЙ ТЕЧЕНИЯ И ТЕМПЕРАТУРЫ, ОКРУЖАЮЩИХ ПУЗЫРЕК ГАЗА И ПАРА

Аннотация — Метод глобальных коллокаций использовался для потенциальных и вязких полей скорости течения вокруг пузырька. Когда пузырек газа является телом вращения, исследовались образование струи, начальное ускорение, отрыв от горизонтальной стенки и движение, вызванное градиентом поверхностного натяжения. Для сферического пузырька пара рассматривались разрушение и рост пузырька и найдено, что радиус пузырька и температура непостоянны. Результаты расчета сравнивались с имеющимися экспериментальными данными. Метод глобальных коллокаций особенно удобен для случая пузырька, окруженного бесконечно простирающейся жидкостью.